R-Matrix and K-matrix Analysis of Elastic α-α Scattering

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Abstract

The R- and K-matrix parametrizations are analyzed and compared for the elastic α-α scattering at center-of-mass energies below 40 MeV. The two parametrizations differ in their definitions of the resonance energy which can lead to quite different results. The physical values of the best-fit parameters are compared with those computed for a potential model. The existence of a broad resonance near 9 MeV is not supported by the data or by the potential model. We discuss the positive and negative aspects for both parametrizations.

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I. INTRODUCTION

The astrophysical S-factor for the reaction $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$ at the astrophysically most effective energy of E=0.3 MeV has been obtained from the extrapolation of a parametrized cross section by fitting data in the center of mass energy range between 1 and 3 MeV. Although Azuma *et al.* have recently found R-matrix and K-matrix parametrizations which give nearly the same results [1], this agreement has not always been observed and fits with quite distinct differences between K- and R-matrix have been found [2–9]. This situation has motivated us to make detailed tests of the two parametrizations and to compare the physical values from the two parametrizations. To this end, one must turn to a much simpler problem than the simultaneous parametrization of the three data sets for $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$, $^{12}\text{C}(\alpha, \alpha)^{12}\text{C}$ and ^{16}N β-decay. Following Barker [10] we consider the s-wave $\alpha + \alpha$ elastic scattering. This problem also has the advantage that, besides accurate data, an excellent potential model description [11] is available which we can use as a benchmark. Below the energy of the first reaction threshold, $^{7}\text{Li}+\text{p}$ at 17.3 MeV, the 43 data we use [1] are the same as in reference [10]. But, when it is useful to consider higher energies, we supplement them by the real parts of 5 complex phase shifts obtained by Darriulat *et al.* [12], as has been done in [11], in the energy range 26-40 MeV. In the present paper, all energies refer to the center-of-mass system.

The phase shifts corresponding to the potential model are obtained from the radial wave function $u(r, E)$ which solves the radial Schrödinger equation

$$\left[\frac{d^2}{dr^2} - \frac{2M}{\hbar^2} \left(V_N + V_C - E \right) \right] u(r, E) = 0 \tag{1}$$

subject to the boundary condition

$$u(r = 0, E) = 0 \quad . \tag{2}$$

Here the nuclear and Coulomb potentials have the form [11]

$$V_N(r) = V_0 \exp\left(-br^2 \right) \quad ; \tag{3}$$

$$V_C(r) = 4e^2 \text{erf}\left(\beta r \right) / r \quad , \tag{4}$$

respectively. The best fit to the data is obtained with $b = 0.212$ fm^{-2} and $\beta = 0.75$ fm^{-1}. The depth V_0 will be chosen in such a way that the R-matrix and K-matrix parametrizations have a pole at the energy of the ^{8}Be ground state, i.e. at the energy $E_g = 92.08$ keV [10].

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[1] Kindly communicated by F.C. Barker.

II. R-MATRIX FITS

In terms of the R-matrix [13], the s-wave phase shifts are

$$\delta(E) = -\phi(a, E) + \arctan\left(\frac{P(a, E)}{R^{-1} + B - S(a, E)}\right) \quad , \tag{5}$$

where a is the channel radius, B the real boundary condition constant, and

$$\phi(a, E) = \arctan(\frac{F_0}{G_0}) \quad , \tag{6}$$

$$P(a, E) = \frac{\rho}{(F_0^2 + G_0^2)} \quad , \tag{7}$$

$$S(a, E) = P(a, E) \ (F_0 F_0' + G_0 G_0'). \tag{8}$$

$-\phi$, P, and S are the hardsphere phase shift, the penetration factor and the shift function, respectively. The radial Coulomb wave functions F_0, G_0 depend on

$$\rho = kr = 0.309428 r E^{1/2}; \eta = 0.891132 E^{-1/2} \quad , \tag{9}$$

where r is in fm and E is in MeV, while in Eq. (8) the primes stand for $d/d\rho$. For the sake of comparison we adopt the radius parameter $a = 6$ fm from Ref. [10]. (We note that this value appears to be somewhat large. For example, Fowler *et al.* used $a = 4$ fm [15] and the conventional value is $a = 1.44(A_1^{1/3} + A_2^{1/3}) = 4.57$ fm. On the other hand, with $a = 6$ fm, the phase shifts have not quite reached their asymptotic values, as we will see below.)

In terms of the radial factor $u(r, E)$, the R-matrix is

$$R = \frac{u(a, E)}{[a(du/dr)_{r=a} - Bu(a, E)]} \quad . \tag{10}$$

Let us call $E_1, E_2, ...$ the eigenenergies satisfying the boundary condition

$$a(du/dr)_{r=a} - Bu(a, E) = 0 \quad . \tag{11}$$

Together with (1) and (2) this defines a classical Sturm-Liouville problem [14] whose eigenvalues are all real. The R-matrix has been defined in such a way that it has poles at the B-dependent eigenenergies E_i. Its expansion then reads

$$R = \sum_{i=1}^{\infty} \frac{\gamma_i^2}{(E_i - E)} \quad , \tag{12}$$

where the γ_i^2 are the formal [13] reduced widths, in terms of which the formal widths are

$$\Gamma_i = 2\gamma_i^2 P(a, E) \quad . \tag{13}$$

Note that the E_i are not resonance energies. However, in R-matrix theory a resonance is associated with each eigenenergy. Let us first consider the case of a one-pole approximation, $R = \gamma_i^2/(E_i - E)$. We then have

$$\delta + \phi = \arctan\left(\frac{\Gamma_i/2}{E_i + \Delta_i(a, E) - E}\right) \quad, \tag{14}$$

where the energy shift is

$$\Delta_i(a, E) = \gamma_i^2 [B - S(a, E)] \quad . \tag{15}$$

The resonance energy $E_{i,r}$ is defined as the shifted E_i satisfying the equation [13]

$$E_i + \Delta_i(a, E_{i,r}) - E_{i,r} = 0 \quad . \tag{16}$$

Assuming that in the neighborhood of $E_{i,r}$ a linear approximation of $\Delta_i(a, E)$ is satisfactory, the so-called observed [13] reduced width and observed width are

$$(\gamma_i^0)^2 = \frac{\gamma_i^2}{[1 + \gamma_i^2(dS/dE)_{E=E_{i,r}}]} \quad, \tag{17}$$

$$\Gamma_i^0 = 2(\gamma_i^0)^2 P(a, E) \quad, \tag{18}$$

and we have

$$\delta + \phi = \arctan\frac{\Gamma_i^0/2}{E_{i,r} - E} \quad . \tag{19}$$

For the general theory following from Eqs. (5) and (10) we have

$$\delta(E) + \phi(a, E) = \arctan\left[P(a, E)Q(a, E)\right] \tag{20}$$

with

$$Q(a, E) = \frac{1}{(R^{-1} + B - S)} = \frac{u(a, E)}{a(du/dr)_{r=a} - S(a, E)u(a, E)} \quad . \tag{21}$$

The function $Q(a, E)$ is neither an R-function in the sense of Wigner[2] nor a meromorphic function of E. As a generalization of Eq. (16), the resonances are defined as the real energies at which

$$R^{-1} + B - S = 0 \quad, \tag{22}$$

and we will call them $E_j^{(j)}, j = 1, 2, ...$ They are pole energies of $Q(a, E)$ and

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[2] See Ref. [13] p. 277 and the references there to E.P. Wigner's original papers on the mathematical R-functions.

$$\delta(E_j^{(j)}) + \phi(a, E_j^{(j)}) = 90°(\text{mod} \ 180°) \ . \tag{23}$$

The pole term of Q corresponding to $E_i^{(i)}$ is

$$\frac{(\gamma_i^0)^2}{(E_i^{(i)} - E)} \tag{24}$$

with

$$(\gamma_i^0)^2 = -\left[(\frac{d}{dE}Q^{-1})_{E=E_i^{(i)}}\right]^{-1} \ . \tag{25}$$

The $E_{i,r}$ and $(\gamma_i^0)^2$ defined by Eqs. (16) and (17) are one-level approximations of the generalized quantities $E_i^{(i)}$ and $(\gamma_i^0)^2$ defined by Eqs. (22) and (25). When a good potential model is available, Q is given by Eq. (21). However, generally this is not the case. Provided an R-matrix fit has been obtained to the data, one can obtain the best values for $E_i^{(i)}$ and $(\gamma_i^0)^2$ by substituting the parametrized R-function into Eqs. (20) to (25).

When fitting data it is often convenient to choose the constant B so that the quantity $B - S(a, E)$ vanishes in Eq. (5) at one of the energies $E_i^{(i)}$, say at $E_k^{(k)}$. When $B = S(a, E_k^{(k)})$, we will call the eigenenergies which satisfy Eq. (11) by $E_1^{(k)}, E_2^{(k)}, ...$

Turning to our example case, elastic $\alpha + \alpha$ scattering, we choose $E_1^{(1)} = E_g$ and $B = S(6 \ \text{fm}, E_g)$. Then the boundary condition (11) is satisfied at $E = E_1^{(1)}$, if the potential depth is $V_0 = -119.216148$ MeV. Since $\phi(a = 6 \ \text{fm}, E_g)$ is very small (equal to $0.00029°$), we have to a very good approximation

$$\delta(E_g) = 90° \ ; (d\delta/dE)_{E=E_g} > 0 \tag{26}$$

and E_g practically coincides with a resonance energy as defined by the conventional definition.

In Fig. 1 the phase shifts obtained with the potential model are compared with the data. How do the poles in the best-fit R-matrix compare with those of the potential model? To answer this question we have computed the energies $E_i^{(j)}$, the reduced and the formal widths for $i = 1 - 3, j = 1 - 3$ and the observed Γ_i^0 widths for $i = 1 - 3$. The results are in Table 1, while we have summarized the best-fit R-matrix parametrizations to the data below 18 MeV in Table 2, assuming $B = S(a = 6 \ \text{fm}, E_j^{(j)})$ with $j = 1 - 3$. No widths can be attributed to $E_1^{(2)}$ and $E_1^{(3)}$, which are below threshold. But it is easily verified that, using the one-pole approximation with Eq. (16), $E_1^{(2)} = -0.395$ MeV in Table 2 is shifted to $E_{1,r} = 0.101$ MeV, close to the resonance energy, while using Eq. (22), $E_1^{(2)}$ is shifted exactly to the resonance energy E_g. Similar results hold for $E_1^{(3)} = -0.379$ MeV. In table 2, the three χ^2 are different. This might be surprising at first glance, as a transformation of the R-matrix parameters from one boundary condition to another should not change the quality of the fits. However, this is only true if *all* R-matrix parameters are allowed to vary [17], which is not the case here.

We also used the results in Table 1 to calculate the R-matrix phase shifts with a 3-pole approximation for the case $B = S(a = 6 \ \text{fm}, E_1^{(1)})$. In Fig. 2, these calculated phase shifts

are compared with the exact potential model phase shifts. The agreement is not good. Comparing Tables 1 and 2, one observes that the reduced widths γ_3^2 of the third poles in Table 2 (2.112, 1.892, and 1.821 MeV) are much larger than those in Table 1 (0.734, 0.769, and 0.770 MeV). Clearly, in the best fits (in Table 2), the large reduced widths at the third poles compensate the contributions from poles at much higher energies. The convergence of the sum of poles calculated for the potential model is very slow. Even adding the fourth and fifth poles (at $E_4^{(1)} = 56.0$ MeV and $E_5^{(1)} = 89.2$ MeV with the reduced widths 0.688 and 0.660 MeV, respectively) is not sufficient for a good fit, as seen in Fig. 2.

Like $Q(a, E)$, the observed resonance energies $E_j^{(j)}$ and the corresponding observed reduced widths $(\gamma_j^0)^2$ are B-independent, but they also depend strongly on the radius a. This is illustrated in Table 3 and Fig. 3 for the potential model. In Fig. 3, with different channel radii, we plotted the phase $\delta + \phi$ corresponding to the data and to the potential model. The best fit of the potential model to the 48 data is obtained with $a = 5.5$ fm and $\chi^2 = 84.1$. For each radius the potential depth V_0 has been modified to satisfy exactly $E_1^{(1)} = E_g$. In the range $a = 7 - 8$ fm, the asymptotic phase has been reached and the fit remains good ($\chi^2 \le 100$). The density of resonances increases strongly with the channel radius without greatly changing χ^2.

The uncertainty regarding the "proper" choice of the channel radius is such that, in practice, it is often chosen to give the best fit to the data.

III. K-MATRIX FITS

For the potential model the conventional K-matrix reads

$$\mathrm{K} = -\left(\frac{uF_0' - u'F_0}{uG_0' - u'G_0}\right)_{r=a} \tag{27}$$

and the phase shifts are

$$\delta(E) = \arctan\mathrm{K} \quad . \tag{28}$$

With the channel radius $a = 6$ fm and the potential depth [3] $V_0 = -119.217576$ MeV, the K-matrix has a pole at $e_1 = E_g = 92.08$ keV, while other real poles are at $e_2 = 2.808641$ MeV and at $e_3 = 31.881031$ MeV. But K cannot be parametrized as a sum of pole terms, because of the essential singularities of the radial Coulomb wave functions F_0, G_0. The K-matrix has an infinite number of complex poles converging to $E = 0$.

To eliminate these singularities, a modified $\mathcal{K}$-matrix has been defined in Ref. [16]. It has no other singularities than isolated poles. It is obtained by separating the threshold factors of F_0, G_0 which depend only on η, and substituting into G_0 a polynomial in η^{-2} (i.e. in E) to the singular function

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[3] The potential depths V_0 are not the same in the R- and K-matrix parametrizations. However, because the energy E_g is very small we have $|F_0| << |G_0|$ and $S(a, E_g) \approx (\rho G_0'/G_0)$, so that the two depths are nearly the same.

$$h(\eta) = \frac{1}{2}\psi(1 + i\eta) + \frac{1}{2}\psi(1 - i\eta) - \ln\eta \quad , \tag{29}$$

where ψ is the digamma function. The function $h(\eta)$ has an essential singularity at $\eta = \infty$, i.e. at $E = 0$. At real energies, over a finite range (say for E between r_1 and r_2), the polynomial is chosen to fit the function $h(\eta)$ for $E \geq 0$ and the function

$$h^+(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \ln(i\eta) \tag{30}$$

for $E \leq 0$. This is easily achieved, at any desired approximation, using Chebyshev polynomials up to the appropriate degree in η^{-2}. Let n be the maximum degree chosen for these polynomials. For a given n, one can then choose r_1 and r_2 so that the polynomial in η^{-2} is exactly equal to $h(\eta)$ at $E = e_1$ and e_2. With $n = 95$ and

$$r_1 = -4.874644 \text{MeV} \; ; \; r_2 = 44.591714 \text{MeV} \tag{31}$$

the real poles e_1, e_2 are the same for K and $\mathcal{K}$, and the e_3 pole is only very slightly shifted to $e_3 = 31.881027$ MeV.

The energy e_1 of the first pole satisfies the usual resonance conditions

$$\delta(e_i) = 90°(\text{mod} \;\; 180°) \, , \;\; (\frac{d\delta}{dE})_{E=e_i} > 0 \tag{32}$$

and the reduced width g_1^2 is positive (see Table 4). For the two other poles, the energies e_2, and e_3 satisfy only the first condition. Their reduced widths are negative. They are echo poles and contribute only to the background of $\mathcal{K}$.

The $\mathcal{K}$-matrix corresponding to the potential model is the meromorphic function

$$\mathcal{K} = -\left(\frac{u\bar{F}_0' - u'\bar{F}_0}{u\bar{G}_0' - u'\bar{G}_0} \right)_{r=a} \quad , \tag{33}$$

where $\bar{F}_0, \bar{G}_0$ are the modified Coulomb wave functions [16], while for the phase shifts, we have

$$\delta(E) = \arctan(p^2\mathcal{K}) \tag{34}$$

with [4]

$$p^2 = \frac{2\pi}{(\exp\{2\pi\eta\} - 1)} \tag{35}$$

The very way we have defined the $\mathcal{K}$ matrix has several consequences. Contrary to K, $\mathcal{K}$ can be expanded in a series of pole terms

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[4] In order to have dimensionless $\mathcal{K}$ and reduced widths with an energy dimension, we substitute $\eta^{-l-1/2}$ for $k^{l+1/2}$ following the definition of p by Eq. (4.7) in Ref. [16].

$$\mathcal{K} = \sum_{i=1}^{\infty} \frac{g_i^2}{(e_i - E)} \quad . \tag{36}$$

At positive real energies, the phase shifts defined by Eqs. (28) and (34) are practically equal, while, at complex energies, $\mathcal{K}$ is different from K/p^2. Like the polynomial in η^{-2}, $\mathcal{K}$ also depends on the choice of n, r_1, and r_2.

With the potential model, the three real poles of $\mathcal{K}$ below 40 MeV and their reduced widths are given in Table 4. With the corresponding 3-pole approximation of $\mathcal{K}$, one does not obtain a good fit of the phase shifts. A better approximation requires more real and/or complex poles. Like the expansion of the R-matrix, the expansion of $\mathcal{K}$ converges only very slowly, and the existence of complex poles introduces further complications. Moreover, only the real poles are uniquely defined, in the sense that the energy of the complex poles and their residues depend on the choice of n, r_1 and r_2. In particular, the density of complex poles near the real axis increases when a larger n is chosen.

In the domain $-4 \text{ MeV} \leq \text{Re}E \leq 44 \text{ MeV}, -5 \text{ MeV} \leq \text{Im}E \leq 5 \text{ MeV}$ with $n = 45, 70, 95$ the number of pairs of complex conjugate poles is 7, 22, and 37, respectively. Many of these poles have very small complex residues and their contributions to the $\mathcal{K}$-matrix are negligible.

In a $\mathcal{K}$-matrix fit to the data, when an additional background term is needed, it cannot be a real pole in the energy range of the data, since $\delta = 90°(\text{mod } 180°)$ only at e_1, e_2, e_3. But a real background pole at negative energy cannot be excluded a priori. In fact, the energy dependence of the $\mathcal{K}$-matrix background plotted in Fig. 2 of [10] shows a decreasing positive background with a concave curvature. Since the energy dependence of an echo pole below threshold has also a concave curvature at positive energies, introducing such a pole into the parametrization seems to be the simplest way to get a good fit below 18 MeV. With the e_1 and e_2 poles and a 3-parameter background composed of a constant and an echo pole at -0.327 MeV, Barker [10] easily obtained a very good fit to the data. We checked that the $\mathcal{K}$ matrix for the potential model has no such pole at negative energies and we must conclude that the three parameters of the background are only *ad hoc* parameters without physical meaning.

To further confirm this conjecture, we made another best fit to the data up to 40 MeV using the three real poles and a pair of complex conjugate poles. With $e_4 = e_R + ie_I$ and $g_4^2 = \gamma_R + i\gamma_I$, the background term has the form

$$\frac{g_4^2}{2(e_4 - E)} + \text{conj.} = \frac{\gamma_R(e_R - E) + \gamma_I e_I}{(e_R - E)^2 + e_I^2} \quad . \tag{37}$$

A very good fit is obtained to the 48 data points up to 40 MeV ($\chi^2 = 27.0$). The results are in Table 4 and Fig. 4. They confirm that in a $\mathcal{K}$ matrix fit to data with a minimal number of parameters, the background parameters are unlikely to correspond to the complex poles computed for a phase-equivalent potential model.

IV. CONCLUSIONS

The main advantage of an R-matrix parametrization is that R can be expanded in terms of real pole terms because R is an R-function in the sense of Wigner. The parameters

introduced in an R-matrix fit are the eigenenergies E_i and the formal reduced widths γ_i^2. However, this is a parametrization of $\delta + \phi$ rather than of δ. One sees in Fig. 3 that at $E \geq 2$ MeV $\delta + \phi$ is steadily increasing with E and one can obtain the energies, at which

$$\delta(E) + \phi(a, E) = 90°(\text{mod } 180°) \quad , \tag{38}$$

defining the so-called observed R-matrix resonance energies. The corresponding observed reduced widths $(\gamma_i^0)^2$ and the observed widths Γ_i^0 are obtained from Eqs. (16) and (17), or, if a 1-pole approximation does not apply, from Eqs. (22) and (25).

Do all these so-called resonance energies $E_i^{(i)}$ really correspond to physical resonances? How are they compared to the resonances defined by the usual conditions

$$\delta(E) = 90°(\text{mod } 180°), \frac{d\delta}{dE} > 0 \quad ? \tag{39}$$

Let us first evaluate the time delay in the scattering process, which is $2v^{-1}d\delta/dk$ [13], where $v = \hbar k/M$ (M being the reduced mass) and

$$\frac{d\delta}{dk} = \frac{d}{dk}\left[\arctan(P(a, E)Q(a, E))\right] - \frac{d}{dk}\phi(a, E) \quad . \tag{40}$$

With the potential model, at $E = E_1^{(1)}$, the two terms on the right hand side of Eq. (40) are $61 \cdot 10^4$ and $97 \cdot 10^{-5}$ fm, respectively, and the resonance conditions (39) are satisfied. At $E = E_2^{(2)}$, the same terms are 3.7 and 6.6 fm, respectively. Thus, $d\delta/dk$ is negative and this corresponds to a time advance, excluding a physical resonance. A similar argument holds also at $E = E_i^{(i)}$ when $i > 2$, but in Ref. [10], the third pole at 32.9 MeV was already considered as a background term.

Let us now see how the $E_1^{(2)}$ R-matrix pole appears as a complex pole of the $\mathcal{S}$-matrix. Designating that pole by $p(E)$, we obtained

$$p(E) = \frac{(-0.0535 + i0.0339)}{(E_c - E)} \tag{41}$$

where $E_c = 3.34 - i16.59$ MeV is far off the real axis. With $\mathcal{S}(3.34) = -0.921 + i0.389$ and $p(3.34) = -0.00204 - i0.00322$, one finds that $p(E)$ does not add a resonant contribution to $\mathcal{S}(E)$.

Under such conditions, we must conclude that, in the best fit, $E_2^{(1)}$ and $E_3^{(1)}$ are both background poles, despite the fact that γ_2^2 and γ_3^2 (like γ_1^2) are positive. This has been confusing [10] and, like the strong dependence of the $E_i^{(i)}$ on the channel radius, it is a weak point of the R-matrix parametrization. This does not concern the good quality of fits one can obtain with this method, but the physical interpretation of the parameters of the pole terms.

In a $\mathcal{K}$ matrix parametrization, the approximate energies at which there are real poles of $\mathcal{K}$ are directly suggested by the data, since they are the energies at which $\delta = 90°(\text{mod } 180°)$. This is the main advantage of a $\mathcal{K}$-matrix parametrization. These energies are either resonances or echo pole energies according to whether the phase shift data are increasing or decreasing at the energies concerned. In the present case, the echo poles at 2.8 MeV and

31.8 MeV are only a part of the background terms of $\mathcal{K}$. At least one more background term is needed to obtain a good fit to the data. It can be a constant, a real echo pole below threshold or at higher energies than the data, or a pair of conjugate complex energy poles. The parameters of these poles, energies and residues, are no more than *ad hoc* parameters. They are not expected to be poles of the modified $\mathcal{K}$-matrix corresponding to a potential model. This is the weak point of a $\mathcal{K}$-matrix parametrization.

We can summarize as follows the lessons we have learned from the present analysis of the $\alpha + \alpha$ scattering using the R-matrix and $\mathcal{K}$-matrix methods for pratical fit procedures. In order to obtain a good fit to data, one should use as flexible parametrizations as possible, without worrying about the relation of certain parameters (e.g. echo poles below threshold in the $\mathcal{K}$-matrix or R-matrix eigenenergies and residues) to physical quantities (e.g. resonance parameters). In certain cases such a connection cannot be made and some of the fit parameters are purely *ad hoc*. Nevertheless, one of the main purposes of R-matrix and $\mathcal{K}$-matrix fits is to give a reasonable and physically motivated basis to extrapolate data. If both methods are used carefully, they can be expected to give similarly good parametrizations, as in the case of $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$.

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FIGURES

FIG. 1. Comparison of the experimental phase shifts (points) with the potential model (solid line). The dashed line corresponds to the best R-matrix fits in Table 2. Note that the 3 best fits are indistinguishable within the thickness of the line.

FIG. 2. Comparison of the potential model phase shifts (solid line) with the computed 3-pole (long-dashed) and 5-pole (short-dashed) R-matrix approximations. Compared with the data, the χ^2 values are 93.9 (potential model), 9202 (3-pole) and 3897 (5-pole), respectively.

FIG. 3. Values for $\delta + \phi$ for the data and the potential model to illustrate the dependence of the R-matrix $E_i^{(i)}$ resonances on the channel radius.

FIG. 4. $\mathcal{K}$-matrix fits to the data. The solid curve (fitted to the data up to 40 MeV) has been obtained with a pair of conjugate complex poles as background, and the dashed line (fitted to the data up to 20 MeV) with a background consisting of a constant and an echo pole below threshold.

TABLES

TABLE I. With the potential-model, three different computed parametrizations of the R-matrix are obtained with the boundary-condition constants $B = S(6, E_i^{(i)})$ $(i = 1, 2, 3)$, respectively. All the parameters are in MeV.

i	1	2	3	4	5
$E_i^{(1)}$	0.09208	9.764	29.274	56.006	89.217
γ_i^2	0.228	0.835	0.734	0.688	0.660
Γ_i	8.15E–6	9.207	14.500	18.947	23.022
$E_i^{(2)}$	–0.343	8.419	28.173		
γ_i^2	0.386	1.004	0.769		
Γ_i	—	10.195	14.903		
$E_i^{(3)}$	–0.352	8.397	28.156		
γ_i^2	0.389	1.006	0.770		
Γ_i	—	10.203	14.904		
$(\gamma_i^0)^2$	0.174	1.000	0.769		
Γ_i^0	6.20E–6	10.156	14.901		

TABLE II. Best R-matrix fits below 18 MeV. The parameters in parentheses denote fixed input values. The fixed parameters for the second and third best fits are obtained using Eq. (21) with the parametrized R-matrix from the first best fit. All the parameters are in MeV. The three χ^2 are not identical because not all parameters have been varied [17].

i	1	2	3	χ^2
$E_i^{(1)}$	(0.09208)	9.787	32.954	
γ_i^2	(0.199)	0.848	2.112	
Γ_i	(7.099E–6)	9.366	44.359	18.05
$E_i^{(2)}$	–0.395	(8.424)	30.463	
γ_i^2	0.343	(1.012)	1.892	
Γ_i	—	(10.281)	38.162	15.86
$E_i^{(3)}$	–0.379	8.406	(30.032)	
γ_i^2	0.336	1.010	(1.821)	
Γ_i	—	10.243	(36.459)	15.90
$(\gamma_i^0)^2$	(0.156)	(1.008)	(1.820)	
Γ_i^0	(5.57E–6)	(10.241)	(36.441)	

TABLE III. Nuclear potential depths, and second and third poles of $Q(a, E)$, for different values of a (in fm). The depths V_0 (in MeV) are chosen so that $E_1^{(1)} = E_g$ for each radius a. V_0 and the $E_i^{(i)}$ are in MeV. The χ^2-values have been calculated for 48 data points.

a	V_0	$E_2^{(2)}$	$E_3^{(3)}$	χ^2
4.0	−122.8034	28.755	78.050	1464
5.0	−119.5106	14.801	44.871	143
5.5	−119.2750	11.018	35.186	84.1
6.0	−119.2161	8.418	28.155	93.9
7.0	−119.2003	5.279	18.980	100
8.0	−119.1999	3.595	13.516	99.7

TABLE IV. $\mathcal{K}$-matrix parameters computed from the potential-model and two best fits up to 18 and 40 MeV, respectively. Except for the dimensionless constant in the first best fit, all the parameters are in MeV.

	Potential model		Best fit		Best fit	
Range	0.01–40 MeV		0.01–18 MeV		0.01–40 MeV	
j	e_j	g_j^2	e_j	g_j^2	e_j	g_j^2
1	0.09208	46.138	(0.09208)	(45.726)	(0.09208)	(45.726)
2	2.809	−10.801	2.819	−10.278	2.809	−10.924
3	31.881	−6.364	−0.324	−39.504	31.865	−5.236
4			const.=−0.656		−1.026±i0.199	−31.082±i307.53
Γ_1	5.62E–6		(5.57E–6)		(5.57E–6)	
χ^2	92.5		18.2		27.0	